# SW(2007) 完全推导与结果复现

夏汉林[1]

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[1] 西南财经大学中国金融研究研究
xiahanlin1993@163.com

# 目 录

# 1 基本模型的设定与求解

# 2 模型对数线性化

## 2.1 SW(1): 资源约束方程的对数线性化

从方程（39）去趋势的资源约束方程 $c_t + i_t + \mathrm{e}^{\varepsilon_t^g} g_y y + \frac{a(Z_t)}{\gamma} k_{t-1} = y_t$ 出发，首先，将式子取全微分可得：

$$\mathrm{d}y_t = \mathrm{d}c_t + \mathrm{d}i_t + g_y y \mathrm{d}\mathrm{e}^{\varepsilon_t^g} + \frac{a(Z_t)}{\gamma}\mathrm{d}k_{t-1} + \frac{k_{t-1}}{\gamma}a'(Z_t)\mathrm{d}Z_t$$

上式在稳态处取值，注意，$Z_t$ 的稳态为 1，$a'(Z_t)$ 的稳态为 $r^k$，$a(Z_t)$ 的稳态为 $a(1) = 0$，因此上式可变为：

$$\begin{aligned}\frac{\mathrm{d}y_t}{y} &= \frac{\mathrm{d}c_t + \mathrm{d}i_t + g_y y \mathrm{d}\mathrm{e}^{\varepsilon_t^g} + \frac{a(Z_t)}{\gamma}\mathrm{d}k_{t-1} + \frac{k_{t-1}}{\gamma}a'(Z_t)\mathrm{d}Z_t}{y} \\ &= \frac{c}{y}\frac{\mathrm{d}c_t}{c} + \frac{i}{y}\frac{\mathrm{d}i_t}{i} + g_y\varepsilon_t^g + \frac{a(1)}{\gamma}\frac{\mathrm{d}k_{t-1}}{y} + \frac{k}{\gamma}\frac{a'(1)}{y}\frac{\mathrm{d}Z_t}{Z} \\ &= \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{r^k}{\gamma}\frac{k}{y}\hat{z}_t + g_y\varepsilon_t^g\end{aligned}$$

得到 SW(1) 的表达式：

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{r^k}{\gamma}\frac{k}{y}\hat{z}_t + \varepsilon_t^g$$

注意，此处将冲击项的系数标准化为 1。

## 2.2 SW(2): 消费一阶条件的对数线性化

从方程（30）去趋势的消费一阶条件 $\zeta_t = \left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)^{-\sigma_c} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}_t^{1+\sigma_l}}$ 和方程（32）去趋势的债券一阶条件 $\zeta_t = \beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}R_t E_t\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]$ 出发：

1. 对去趋势的消费一阶条件 $\zeta_t = \left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)^{-\sigma_c} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}_t^{1+\sigma_l}}$ 取全微分：

$$\begin{aligned}\mathrm{d}\zeta_t = -\sigma_c\left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)^{-\sigma_c - 1} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}_t^{1+\sigma_l}}\mathrm{d}c_t + \sigma_c\frac{\lambda}{\gamma}\left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)^{-\sigma_c - 1} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}_t^{1+\sigma_l}}\mathrm{d}c_{t-1} \\ + \left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)^{-\sigma_c} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}_t^{1+\sigma_l}}\frac{\sigma_c - 1}{1+\sigma_l}(1+\sigma_l)\bar{L}_t^{\sigma_l}\mathrm{d}\bar{L}_t\end{aligned}$$

2. 对除 d 内的其他内容在稳态处取值：

$$\begin{aligned}\mathrm{d}\zeta_t = -\sigma_c\left(c_- \frac{\lambda}{\gamma}c\right)^{-\sigma_c - 1} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}^{1+\sigma_l}}\mathrm{d}c_t + \sigma_c\frac{\lambda}{\gamma}\left(c - \frac{\lambda}{\gamma}c\right)^{-\sigma_c - 1} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}^{1+\sigma_l}}\mathrm{d}c_{t-1} \\ + \left(c - \frac{\lambda}{\gamma}c\right)^{-\sigma_c} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}^{1+\sigma_l}}(\sigma_c - 1)\bar{L}^{\sigma_l}\mathrm{d}\bar{L}_t\end{aligned}$$

消费一阶条件 $\zeta_t = \left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)^{-\sigma_c} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}_t^{1+\sigma_l}}$ 在稳态处取值：

$$\zeta = \left(c - \frac{\lambda}{\gamma}c\right)^{-\sigma_c} \mathrm{e}^{\frac{\sigma_c - 1}{1+\sigma_l}\bar{L}^{1+\sigma_l}}$$

$$\frac{\mathrm{d}\zeta_t}{\zeta}=-\sigma_c\left(1-\frac{\lambda}{\gamma}\right)^{-1}\frac{\mathrm{d}c_t}{c}+\sigma_c\frac{\lambda}{\gamma}\left(1-\frac{\lambda}{\gamma}\right)^{-1}\frac{\mathrm{d}c_{t-1}}{c}+(\sigma_c-1)\bar{L}^{\sigma_l}\mathrm{d}\bar{L}_t$$
$$=-\sigma_c\frac{1}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_t+\sigma_c\frac{\frac{\lambda}{\gamma}}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_{t-1}+(\sigma_c-1)\bar{L}^{\sigma_l+1}\hat{l}_t$$

3. 对方程（32）去趋势的债券一阶条件 $\zeta_t=\beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}R_tE_t\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]$ 取全微分：

$$\mathrm{d}\zeta_t=\beta\gamma^{-\sigma_c}R_tE_t\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]\mathrm{d}\mathrm{e}^{\varepsilon_t^b}+\beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]\mathrm{d}R_t+\beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}R_tE_t\left[\frac{1}{\pi_{t+1}}\mathrm{d}\zeta_{t+1}-\frac{\zeta_{t+1}}{\pi_{t+1}^2}\mathrm{d}\pi_{t+1}\right]$$

4. 对除 d 内的其他内容在稳态处取值：

$$\mathrm{d}\zeta_t=\beta\gamma^{-\sigma_c}R\frac{\zeta}{\pi}\mathrm{d}\mathrm{e}^{\varepsilon_t^b}+\beta\gamma^{-\sigma_c}\frac{\zeta}{\pi}\mathrm{d}R_t+\beta\gamma^{-\sigma_c}RE_t\left[\frac{1}{\pi}\mathrm{d}\zeta_{t+1}-\frac{\zeta}{\pi^2}\mathrm{d}\pi_{t+1}\right]$$

债券一阶条件 $\zeta_t=\beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}R_tE_t\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]$ 在稳态处取值：

$$\zeta=\beta\gamma^{-\sigma_c}R\frac{\zeta}{\pi}$$

$$\frac{\mathrm{d}\zeta_t}{\zeta}=\mathrm{d}\mathrm{e}^{\varepsilon_t^b}+\frac{\mathrm{d}R_t}{R}+E_t\left[\frac{\mathrm{d}\zeta_{t+1}}{\zeta}-\frac{\mathrm{d}\pi_{t+1}}{\pi}\right]=\varepsilon_t^b+\hat{r}_t+E_t\hat{\zeta}_{t+1}-E_t\hat{\pi}_{t+1}$$

将 $\hat{\zeta}_t=-\sigma_c\frac{1}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_t+\sigma_c\frac{\frac{\lambda}{\gamma}}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_{t-1}+(\sigma_c-1)\bar{L}^{\sigma_l+1}\mathrm{d}\hat{l}_t$ 代入上式可得：

$$-\sigma_c\frac{1}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_t+\sigma_c\frac{\frac{\lambda}{\gamma}}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_{t-1}+(\sigma_c-1)\bar{L}^{\sigma_l+1}\hat{l}_t$$
$$=-\sigma_c\frac{1}{\left(1-\frac{\lambda}{\gamma}\right)}E_t\hat{c}_{t+1}+\sigma_c\frac{\frac{\lambda}{\gamma}}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_t+(\sigma_c-1)\bar{L}^{\sigma_l+1}\hat{l}_{t+1}+\varepsilon_t^b+\hat{r}_t-E_t\hat{\pi}_{t+1}$$

合并关于 $\hat{c}_t$ 的相关项：

$$-\sigma_c\frac{1+\frac{\lambda}{\gamma}}{1-\frac{\lambda}{\gamma}}\hat{c}_t=-\sigma_c\frac{\frac{\lambda}{\gamma}}{\left(1-\frac{\lambda}{\gamma}\right)}\hat{c}_{t-1}-\sigma_c\frac{1}{\left(1-\frac{\lambda}{\gamma}\right)}E_t\hat{c}_{t+1}-(\sigma_c-1)\bar{L}^{\sigma_l+1}(\hat{l}_t-E_t\hat{l}_{t+1})+\varepsilon_t^b+\hat{r}_t-E_t\hat{\pi}_{t+1}$$

两边同除 $-\sigma_c\frac{1+\frac{\lambda}{\gamma}}{1-\frac{\lambda}{\gamma}}$ 可得：

$$\hat{c}_t=\frac{\frac{\lambda}{\gamma}}{1+\frac{\lambda}{\gamma}}\hat{c}_{t-1}+\frac{1}{1+\frac{\lambda}{\gamma}}E_t\hat{c}_{t+1}+\frac{\left(1-\frac{\lambda}{\gamma}\right)(\sigma_c-1)\bar{L}^{\sigma_l+1}}{\sigma_c\left(1+\frac{\lambda}{\gamma}\right)}(\hat{l}_t-E_t\hat{l}_{t+1})-\frac{1-\frac{\lambda}{\gamma}}{\sigma_c\left(1+\frac{\lambda}{\gamma}\right)}(\varepsilon_t^b+\hat{r}_t-E_t\hat{\pi}_{t+1})$$

由方程（31）可得 $w^h=(1-\frac{\lambda}{\gamma})cL^{\sigma_l}$，$\left(1-\frac{\lambda}{\gamma}\right)\bar{L}^{\sigma_l+1}=\frac{w^hL}{c}$，上式最后变为：

$$\hat{c}_t=\frac{\frac{\lambda}{\gamma}}{1+\frac{\lambda}{\gamma}}\hat{c}_{t-1}+\frac{1}{1+\frac{\lambda}{\gamma}}E_t\hat{c}_{t+1}+\frac{(\sigma_c-1)}{\sigma_c\left(1+\frac{\lambda}{\gamma}\right)}\frac{w^hL}{c}(\hat{l}_t-E_t\hat{l}_{t+1})-\frac{1-\frac{\lambda}{\gamma}}{\sigma_c\left(1+\frac{\lambda}{\gamma}\right)}(\hat{r}_t-E_t\hat{\pi}_{t+1}+\varepsilon_t^b)$$

## 2.3 SW(3): 投资一阶条件的对数线性化

从方程(33)去趋势的投资的一阶条件 $1 = Q_t e^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right) - S'\left(\frac{i_t\gamma}{i_{t-1}}\right)\frac{i_t\gamma}{i_{t-1}}\right] + \beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[Q_{t+1}\mathrm{e}^{\varepsilon_{t+1}^i}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2\right]$ 出发:

1. 对方程(33)取全微分:

$$
\begin{aligned}
0 = e^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right) - S'\left(\frac{i_t\gamma}{i_{t-1}}\right)\frac{i_t\gamma}{i_{t-1}}\right]\mathrm{d}Q_t + Q_t\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right) - S'\left(\frac{i_t\gamma}{i_{t-1}}\right)\frac{i_t\gamma}{i_{t-1}}\right]\mathrm{d}e^{\varepsilon_t^i} \\
+Q_t e^{\varepsilon_t^i}\left[\mathrm{d}S\left(\frac{i_t\gamma}{i_{t-1}}\right) - \frac{i_t\gamma}{i_{t-1}}\mathrm{d}S'\left(\frac{i_t\gamma}{i_{t-1}}\right) - S'\left(\frac{i_t\gamma}{i_{t-1}}\right)\mathrm{d}\frac{i_t\gamma}{i_{t-1}}\right] \\
+\beta\gamma^{-\sigma_c}E_t\frac{1}{\zeta_t}\left[Q_{t+1}\mathrm{e}^{\varepsilon_{t+1}^i}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2\right]\mathrm{d}\zeta_{t+1} \\
-\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t^2}\left[Q_{t+1}\mathrm{e}^{\varepsilon_{t+1}^i}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2\right]\mathrm{d}\zeta_t \\
+\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[\mathrm{e}^{\varepsilon_{t+1}^i}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2\right]\mathrm{d}Q_{t+1} \\
+\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[Q_{t+1}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2\right]\mathrm{d}e^{\varepsilon_{t+1}^i} \\
+\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[Q_{t+1}\mathrm{e}^{\varepsilon_{t+1}^i}\left(\frac{i_{t+1}\gamma}{i_t}\right)^2\right]\mathrm{d}S'\left(\frac{i_{t+1}\gamma}{i_t}\right) \\
\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[Q_{t+1}\mathrm{e}^{\varepsilon_{t+1}^i}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\right]\mathrm{d}\left(\frac{i_{t+1}\gamma}{i_t}\right)^2
\end{aligned}
$$

2. 对除 d 内的其他式子取稳态处值:

$$\mathrm{d}S\left(\frac{i_t\gamma}{i_{t-1}}\right) = S'\left(\frac{i\gamma}{i}\right)\gamma\frac{i\mathrm{d}i_t - i\mathrm{d}i_{t-1}}{i^2}$$

由于 $S'(\gamma) = 0$, $\mathrm{d}S\left(\frac{i_t\gamma}{i_{t-1}}\right) = 0$, 上式中所有包含 $\mathrm{d}S\left(\frac{i_t\gamma}{i_{t-1}}\right)$ 的式子在稳态值取值都为 0。包含 $S'(\gamma)$ 的式子取值也为 0。

$$\frac{i_t\gamma}{i_{t-1}}\mathrm{d}S'\left(\frac{i_t\gamma}{i_{t-1}}\right) = \gamma S''\left(\frac{i\gamma}{i}\right)\gamma\frac{i\mathrm{d}i_t - i\mathrm{d}i_{t-1}}{i^2} = \gamma^2 S''(\gamma)(\hat{i}_t - \hat{i}_{t-1})$$

令 $S''(\gamma) = \varphi$, 上式为

$$\frac{i_t\gamma}{i_{t-1}}\mathrm{d}S'\left(\frac{i_t\gamma}{i_{t-1}}\right) = \gamma S''\left(\frac{i\gamma}{i}\right)\gamma\frac{i\mathrm{d}i_t - i\mathrm{d}i_{t-1}}{i^2} = \gamma^2\varphi(\hat{i}_t - \hat{i}_{t-1})$$

用上述结果对全微分式子进行简化:

$$0 = \mathrm{d}Q_t + \mathrm{d}e^{\varepsilon_t^i} - \gamma^2\varphi(\hat{i}_t - \hat{i}_{t-1}) + \beta\gamma^{-\sigma_c}\gamma^3\varphi(\hat{i}_{t+1} - \hat{i}_t) = \hat{q}_t + \varepsilon_t^i - \gamma^2\varphi(\hat{i}_t - \hat{i}_{t-1}) + \beta\gamma^{-\sigma_c}\gamma^3\varphi(\hat{i}_{t+1} - \hat{i}_t)$$

将关于 $\hat{i}_t$ 的项归集到左边：

$$
\begin{aligned}
&\gamma^2\varphi\hat{i}_t+\beta\gamma^{-\sigma_c}\gamma^3\varphi\hat{i}_t=\hat{q}_t+\varepsilon_t^i+\gamma^2\varphi\hat{i}_{t-1}+\beta\gamma^{-\sigma_c}\gamma^3\varphi\hat{i}_{t+1}\\
\Longrightarrow&\gamma^2\varphi(1+\beta\gamma^{1-\sigma_c})\hat{i}_t=\hat{q}_t+\varepsilon_t^i+\gamma^2\varphi\hat{i}_{t-1}+\beta\gamma^{-\sigma_c}\gamma^3\varphi\hat{i}_{t+1}\\
\Longrightarrow&\hat{i}_t=\frac{1}{\gamma^2\varphi(1+\beta\gamma^{1-\sigma_c})}(\hat{q}_t+\varepsilon_t^i)+\frac{1}{1+\beta\gamma^{1-\sigma_c}}\hat{i}_{t-1}+\frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}}\hat{i}_{t+1}\\
\Longrightarrow&\hat{i}_t=\frac{1}{1+\beta\gamma^{1-\sigma_c}}\hat{i}_{t-1}+\frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}}\hat{i}_{t+1}+\frac{1}{\gamma^2\varphi(1+\beta\gamma^{1-\sigma_c})}\hat{q}_t+\varepsilon_t^i
\end{aligned}
$$

注意，此处将冲击项 $\varepsilon_t^i$ 标准化为 1，如果其他地方出现 $\varepsilon_t^i$，应该乘以此处冲击项系数的倒数。

## 2.4 SW(4): 资本价格一阶条件的对数线性化

从方程(34)去趋势的资本存量的一阶条件 $Q_t=\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[r_{t+1}^kZ_{t+1}-a\left(Z_{t+1}\right)+Q_{t+1}(1-\delta)\right]$ 和方程（32）去趋势的债券一阶条件 $\zeta_t=\beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}R_tE_t\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]$ 出发：第一步，对方程（34）进行对数线性化：

1. 对方程（34）取全微分：

$$
\begin{aligned}
\mathrm{d}Q_t=&\beta\gamma^{-\sigma_c}E_t\frac{1}{\zeta_t}\left[r_{t+1}^kZ_{t+1}-a\left(Z_{t+1}\right)+(1-\delta)\right]\mathrm{d}\zeta_{t+1}\\
&-\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t^2}\left[r_{t+1}^kZ_{t+1}-a\left(Z_{t+1}\right)+Q_{t+1}(1-\delta)\right]\mathrm{d}\zeta_t\\
&+\beta\gamma^{-\sigma_c}E_t\frac{\zeta_{t+1}}{\zeta_t}\left[r_{t+1}^k\mathrm{d}Z_{t+1}+Z_{t+1}\mathrm{d}r_{t+1}^k-a'(Z_{t+1})\mathrm{d}Z_{t+1}+(1-\delta)\mathrm{d}Q_{t+1}\right]
\end{aligned}
$$

2. 对除 d 以内的其他式子在稳态处取值：

$$
\begin{aligned}
\mathrm{d}Q_t=&\beta\gamma^{-\sigma_c}\frac{1}{\zeta}\left[r^k+(1-\delta)\right]\mathrm{d}\zeta_{t+1}-\beta\gamma^{-\sigma_c}E_t\frac{1}{\zeta}\left[r^k+(1-\delta)\right]\mathrm{d}\zeta_t\\
&+\beta\gamma^{-\sigma_c}E_t\left[r^k\mathrm{d}Z_{t+1}+\mathrm{d}r_{t+1}^k-r^k\mathrm{d}Z_{t+1}+(1-\delta)\mathrm{d}Q_{t+1}\right]\\
=&\beta\gamma^{-\sigma_c}\left[r^k+(1-\delta)\right]\hat{\zeta}_{t+1}-\beta\gamma^{-\sigma_c}\left[r^k+(1-\delta)\right]\hat{\zeta}_t+\beta\gamma^{-\sigma_c}E_t\left[r^k\hat{r}_{t+1}^k+(1-\delta)\hat{q}_{t+1}\right]\\
=&\hat{\zeta}_{t+1}-\hat{\zeta}_t+\beta\gamma^{-\sigma_c}E_t\left[r^k\hat{r}_{t+1}^k+(1-\delta)\hat{q}_{t+1}\right]
\end{aligned}
$$

注意 $a'(1)=r^k$，在稳态处，有 $\beta\gamma^{-\sigma_c}\frac{1}{\zeta}\left[r^k+(1-\delta)\right]=1$，接下来替换 $\hat{\zeta}_{t+1}-\hat{\zeta}_t$，这需要对方程（32）去趋势的债券一阶条件 $\zeta_t=\beta\gamma^{-\sigma_c}\mathrm{e}^{\varepsilon_t^b}R_tE_t\left[\frac{\zeta_{t+1}}{\pi_{t+1}}\right]$ 进行对数线性化：

1. 对上式取对数

$$\log\zeta_t=\log\beta\gamma^{-\sigma_c}+\varepsilon_t^b+\log R_t+\log\zeta_{t+1}-\log\pi_{t+1}$$

2. 上式取稳态值：

$$\log\zeta=\log\beta\gamma^{-\sigma_c}+\log R+\log\zeta-\log\pi$$

3. 两式相减：

$$
\begin{aligned}
\log\zeta_t-\log\zeta=\varepsilon_t^b+\log R_t-\log R+\log\zeta_{t+1}-\log\zeta-\log\pi_{t+1}-\log\pi\\
\Longrightarrow\hat{\zeta}_t=\hat{r}_t+\hat{\zeta}_{t+1}-\hat{\pi}_{t+1}+\varepsilon_t^b
\end{aligned}
$$

将上式代入 $\hat{q}_t = \hat{\zeta}_{t+1} - \hat{\zeta}_t + \beta\gamma^{-\sigma_c} E_t \left[r^k \hat{r}^k_{t+1} + (1-\delta)\hat{q}_{t+1}\right]$ 得:

$$\begin{aligned}\hat{q}_t &= -(\hat{r}_t - E_t\hat{\pi}_{t+1} + \varepsilon^b_t) + \beta\gamma^{-\sigma_c} E_t \left[r^k \hat{r}^k_{t+1} + (1-\delta)E_t\hat{q}_{t+1}\right] \\ &= \beta\gamma^{-\sigma_c}(1-\delta)E\hat{q}_{t+1} + \beta\gamma^{-\sigma_c} r^k E\hat{r}^k_{t+1} - \left(\hat{r}_t - E\hat{\pi}_{t+1} + \varepsilon^b_t\right)\end{aligned}$$

## 2.5 SW(5): 中间品生产函数的对数线性化

从方程(22) $\int_0^1 G\left(\frac{y_{i,t}}{y_t}; \varepsilon^p_t\right) di = 1$ 和方程(24)去趋势的中间品生产函数 $y_{i,t} = \mathrm{e}^{\varepsilon^a_t}\left(k^s_{i,t}\right)^\alpha (L_{i,t})^{1-\alpha} - \Phi$ 出发:

第一步: 对方程（22）$\int_0^1 G\left(\frac{y_{i,t}}{y_t}; \varepsilon^p_t\right) di = 1$ 取全微分，并取稳态值:

$$\begin{aligned}&\int_0^1 G' \frac{y\mathrm{d}y_{i,t} - y\mathrm{d}y_t}{y^2} di = 0 \\ \Longrightarrow &\int_0^1 \frac{\mathrm{d}y_{i,t} - \mathrm{d}y_t}{y} di = 0 \\ \Longrightarrow &\int_0^1 \frac{\mathrm{d}y_{i,t}}{y} - \frac{\mathrm{d}y_t}{y} di = 0 \\ \Longrightarrow &\int_0^1 \hat{y}_{i,t} di = \int_0^1 \hat{y}_t di \\ \Longrightarrow &\hat{y}_t = \int_0^1 \hat{y}_{i,t} di\end{aligned}$$

第二步: 定义 $\phi_p = \frac{\Phi + y}{y}$, 对方程(24)去趋势的中间品生产函数 $y_{i,t} = \mathrm{e}^{\varepsilon^a_t}\left(k^s_{i,t}\right)^\alpha (L_{i,t})^{1-\alpha} - \Phi$ 进行对数线性化:

1. 对方程（24）取对数:

$$\begin{aligned}&y_{i,t} + \Phi = \mathrm{e}^{\varepsilon^a_t}\left(k^s_{i,t}\right)^\alpha (L_{i,t})^{1-\alpha} \\ \overset{\text{取对数}}{\Longrightarrow} &\log(y_{i,t} + \Phi) = \varepsilon^a_t + \alpha \log k^s_{i,t} + (1-\alpha)\log L_{i,t}\end{aligned}$$

2. 其稳态为

$$\log(y + \Phi) = \alpha \log k^s + (1-\alpha)\log L$$

3. 上面两式相减:

$$\log(y_{i,t} + \Phi) - \log(y + \Phi) = \varepsilon^a_t + \alpha(\log k^s_{i,t} - \log k^s) + (1-\alpha)(\log L_{i,t} - \log L)$$

$$\log \frac{y_{i,t} + \Phi}{y + \Phi} = \log \frac{y_{i,t}}{y} \frac{y}{y + \Phi} = \phi_p^{-1}\hat{y}_{i,t}$$

$$\begin{aligned}&\phi_p^{-1}\hat{y}_{i,t} = \varepsilon^a_t + \alpha(\log k^s_{i,t} - \log k^s) + (1-\alpha)(\log L_{i,t} - \log L) \\ \overset{\text{对 i 在 0-1 上积分}}{\Longrightarrow} &\int_0^1 \phi_p^{-1}\hat{y}_{i,t} di = \int_0^1 \varepsilon^a_t + \alpha(\log k^s_{i,t} - \log k^s) + (1-\alpha)(\log L_{i,t} - \log L) di \\ &\hat{y}_t = \phi_p\left(\alpha\hat{k}^s_t + (1-\alpha)\hat{l}_t + \varepsilon^a_t\right)\end{aligned}$$

## 2.6 SW(6): 资本服务方程的对数线性化

从方程（29）去趋势的资本服务定义方程 $k_t^s = \frac{Z_t k_{t-1}}{\gamma}$ 出发：

1. 对方程（29）取对数：

$$\log k_t^s = \log Z_t + \log k_{t-1} - \log \gamma$$

2. 其稳态处取值为：

$$\log k^s = \log Z + \log k - \log \gamma$$

3. 两式相减：

$$\begin{aligned} \log k_t^s - \log k =& \log Z_t - \log Z + \log k_{t-1} - \log k \\ \Longrightarrow \hat{k}_t^s =& \hat{z}_t + \hat{k}_{t-1} \end{aligned}$$

## 2.7 SW(7): 资本利用率一阶条件的对数线性化

从方程（35）去趋势的资本利用率的一阶条件 $r_t^k = a'(Z_t)$ 出发：

1. 对方程（35）两边取微分：

$$\mathrm{d}r_t^k = a''(1)\mathrm{d}Z_t$$

2. 式（35）在稳态处取值：

$$r^k = a'(1)$$

3. 微分方程除以稳态值：

$$\begin{aligned} \frac{\mathrm{d}r_t^k}{r^k} &= \frac{a''(1)}{a'(1)}\frac{\mathrm{d}Z_t}{Z} \\ \Longrightarrow \hat{r}_t^k &= \frac{a''(1)}{a'(1)}\hat{z}_t \end{aligned}$$

由于 $\frac{a''(1)}{a'(1)} = \frac{\psi}{1-\psi}$，上式可写为：

$$\hat{z}_t = \frac{1-\psi}{\psi}\hat{r}_t^k$$

## 2.8 SW(8): 资本存量一阶条件的对数线性化

从方程（28）去趋势的资本存量的一阶条件 $k_t = \frac{(1-\delta)}{\gamma}k_{t-1} + \mathrm{e}^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right)\right]i_t$ 出发：

1. 对方程（28）取全微分：

$$\begin{aligned} \mathrm{d}k_t = \frac{(1-\delta)}{\gamma}\mathrm{d}k_{t-1} + \mathrm{e}^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right)\right]i_t\mathrm{d}\mathrm{e}^{\varepsilon_t^i} + \mathrm{e}^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right)\right]\mathrm{d}i_t \\ - \mathrm{e}^{\varepsilon_t^i}i_t\mathrm{d}S\left(\frac{i_t\gamma}{i_{t-1}}\right) \end{aligned}$$

2. 除 d 内的其他内容在稳态处取值：

$$
\begin{aligned}
\mathrm{d}k_t &= \frac{(1-\delta)}{\gamma}\mathrm{d}k_{t-1} + \mathrm{e}^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right)\right] i_t \mathrm{d}\mathrm{e}^{\varepsilon_t^i} + \mathrm{e}^{\varepsilon_t^i}\left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right)\right]\mathrm{d}i_t - \mathrm{e}^{\varepsilon_t^i} i_t \mathrm{d}S\left(\frac{i_t\gamma}{i_{t-1}}\right) \\
&= \frac{(1-\delta)}{\gamma}\mathrm{d}k_{t-1} + i\varepsilon_t^i + \mathrm{d}i_t
\end{aligned}
$$

3. 两边同除稳态值 k:

$$
\begin{aligned}
\frac{\mathrm{d}k_t}{k} &= \frac{(1-\delta)}{\gamma}\frac{\mathrm{d}k_{t-1}}{k} + \frac{i}{k}\varepsilon_t^i + \frac{i}{k}\frac{\mathrm{d}i_t}{i} \\
\Longrightarrow \hat{k}_t &= \frac{(1-\delta)}{\gamma}\hat{k}_{t-1} + \left(1 - \frac{1-\delta}{\gamma}\right)\hat{i}_t + \left(1 - \frac{1-\delta}{\gamma}\right)(1+\beta\gamma^{1-\sigma_c})\gamma^2\varphi\varepsilon_t^i
\end{aligned}
$$

注意，由于 $S(\gamma)=0$，所以 $1-S(\gamma)=1$，$S'(\gamma)=0$，方程（28）在稳态处的值 $k=\frac{\gamma}{\gamma-1+\delta}i$，由于 SW(3) 将 $\varepsilon_t^i$ 的系数标准化为 1，此处的 $\varepsilon_t^i$ 需要乘以 $(1+\beta\gamma^{1-\sigma_c})\gamma^2\varphi$，即 SW(3) 冲击项的系数的倒数。

## 2.9 SW(9): 价格 markup 的对数线性化

从方程（26）去趋势的边际成本方程 $mc_t = \frac{w_t}{(1-\alpha)\mathrm{e}^{\varepsilon_t^a}(k_t^s/L_t)^\alpha} = \frac{w_t^{1-\alpha}(r_t^k)^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}\mathrm{e}^{\varepsilon_t^a}}$ 出发，可得价格 Mark-up 方程

1. 对方程（26）取对数：

$$
\log mc_t = \log w_t - \alpha(\log k_t^s - \log L_t) - \varepsilon_t^a
$$

2. 取稳态值：

$$
\log mc = \log w - \alpha(\log k^s - \log L)
$$

3. 两式相减：

$$
\begin{aligned}
\log mc_t - \log mc = \log w_t - \log w - \alpha(\log k_t^s - \log k^s + \log L - \log L_t) - \varepsilon_t^a \\
\Longrightarrow \hat{mc}_t = \hat{w}_t - \alpha(\hat{k}_t^s - \hat{l}_t) - \varepsilon_t^a
\end{aligned}
$$

令 $\mu_t^p = -\hat{mc}_t$，则

$$
\mu_t^p = -\hat{w}_t + \alpha(\hat{k}_t^s - \hat{l}_t) + \varepsilon_t^a
$$

## 2.10 SW(10): 价格菲利普斯曲线的推导

第一步：从方程（23）去趋势的总价格方程 $1=(1-\xi_p)\,\tilde{p}_t G'^{-1}\,(\tilde{p}_t\tau_t^p)+\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}G'^{-1}$ $(\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p)$ 出发，对其进行对数线性化：

1. 两边取全微分:

$$0=(1-\xi_p)\,G'^{-1}\left(\tilde{p}_t\tau_t^p\right)\mathrm{d}\tilde{p}_t+(1-\xi_p)\,\tilde{p}_t\frac{\tau_t^p}{G''\left(x_{it}\right)}\mathrm{d}\tilde{p}_t+(1-\xi_p)\,\tilde{p}_t\frac{\tilde{p}_t}{G''\left(x_{it}\right)}\mathrm{d}\tau_t^p$$
$$+\iota_p\xi_p\pi_{t-1}^{\iota_p-1}\pi^{1-\iota_p}\pi_t^{-1}G'^{-1}\left(\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p\right)\mathrm{d}\pi_{t-1}-\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-2}G'^{-1}\left(\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p\right)\mathrm{d}\pi_t$$
$$+\iota_p\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\frac{\pi_{t-1}^{\iota_p-1}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p}{G''\left(x_{it}\right)}\mathrm{d}\pi_{t-1}-\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\frac{\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-2}\tau_t^p}{G''\left(x_{it}\right)}\mathrm{d}\pi_t$$
$$+\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\frac{\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}}{G''\left(x_{it}\right)}\mathrm{d}\tau_t^p$$

2. 对处 d 内之外的其他表达式在稳态处取值:

   注意，在稳态值处，有 $G'^{-1}(z^*)=x=1$，其中 $x_{i,t}=\frac{y_{i,t}}{y_t}$，$z_{it}=\tilde{p}_t\tau_t^p$，$\tau^p=G'(1)$，$\tilde{p}=1$，因此在稳态值处有:

$$(1-\xi_p)\,G'^{-1}\left(\tilde{p}_t\tau_t^p\right)\mathrm{d}\tilde{p}_t=(1-\xi_p)\,G'^{-1}\left(z^*\right)\mathrm{d}\tilde{p}_t=(1-\xi_p)\,\mathrm{d}\tilde{p}_t=(1-\xi_p)\;\hat{\tilde{p}}_t$$

$$(1-\xi_p)\,\tilde{p}_t\frac{\tau_t^p}{G''\left(x_{it}\right)}\mathrm{d}\tilde{p}_t=(1-\xi_p)\,\frac{G'(1)}{G''(1)}\mathrm{d}\tilde{p}_t=(1-\xi_p)\,\frac{G'(1)}{G''(1)}\,\hat{\tilde{p}}_t$$

$$(1-\xi_p)\,\tilde{p}_t\frac{\tilde{p}_t}{G''\left(x_{it}\right)}\mathrm{d}\tau_t^p=(1-\xi_p)\,\frac{1}{G''\left(1\right)}\mathrm{d}\tau_t^p$$

其中，$\tau_t^p=\int_0^1 G'\left(\frac{y_{i,t}}{y_t};\varepsilon_t^p\right)\frac{y_{i,t}}{y_t}di=\int_0^1 G'\left(x_{i,t};\varepsilon_t^p\right)x_{i,t}di$

$$\begin{aligned}\iota_p\xi_p\pi_{t-1}^{\iota_p-1}\pi^{1-\iota_p}\pi_t^{-1}G'^{-1}\left(\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p\right)\mathrm{d}\pi_{t-1}&=\iota_p\xi_p\pi^{\iota_p-1}\pi^{1-\iota_p}\pi^{-1}G'^{-1}\left(\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-1}\tau^p\right)\\&=\iota_p\xi_pG'^{-1}\left(G'(1)\right)\frac{\mathrm{d}\pi_{t-1}}{\pi}=\iota_p\xi_p\hat{\pi}_{t-1}\end{aligned}$$

注意，$G'^{-1}\left(G'(1)\right)=1$。

$$\begin{aligned}-\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-2}G'^{-1}\left(\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p\right)\mathrm{d}\pi_t&=-\xi_p\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-2}G'^{-1}\left(\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-1}\tau^p\right)\mathrm{d}\pi_t\\&=-\xi_pG'^{-1}\left(G'(1)\right)\frac{\mathrm{d}\pi_t}{\pi}=-\xi_p\hat{\pi}_t\end{aligned}$$

$$\begin{aligned}\iota_p\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\frac{\pi_{t-1}^{\iota_p-1}\pi^{1-\iota_p}\pi_t^{-1}\tau_t^p}{G''\left(x_{it}\right)}\mathrm{d}\pi_{t-1}&=\iota_p\xi_p\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-1}\frac{\pi^{\iota_p-1}\pi^{1-\iota_p}\pi^{-1}\tau^p}{G''\left(1\right)}\mathrm{d}\pi_{t-1}\\&=\iota_p\xi_p\frac{G'(1)}{G''\left(1\right)}\frac{\mathrm{d}\pi_{t-1}}{\pi}=\iota_p\xi_p\frac{G'(1)}{G''\left(1\right)}\hat{\pi}_{t-1}\end{aligned}$$

$$\begin{aligned}-\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\frac{\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-2}\tau_t^p}{G''\left(x_{it}\right)}\mathrm{d}\pi_t&=-\xi_p\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-1}\frac{\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-2}\tau^p}{G''\left(1\right)}\mathrm{d}\pi_t\\&=-\xi_p\frac{G'(1)}{G''\left(1\right)}\frac{\mathrm{d}\pi_t}{\pi}=-\xi_p\frac{G'(1)}{G''\left(1\right)}\hat{\pi}_t\end{aligned}$$

$$\xi_p\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}\frac{\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\pi_t^{-1}}{G''\left(x_{it}\right)}\mathrm{d}\tau_t^p=\xi_p\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-1}\frac{\pi^{\iota_p}\pi^{1-\iota_p}\pi^{-1}}{G''\left(1\right)}\mathrm{d}\tau_t^p=\xi_p\frac{1}{G''\left(1\right)}\mathrm{d}\tau_t^p$$

综合以上式子:

$$0 = (1-\xi_p)\hat{\tilde{p}}_t + (1-\xi_p)\frac{G'(1)}{G''(1)}\hat{\tilde{p}}_t + (1-\xi_p)\frac{1}{G''(1)}\mathrm{d}\tau_t^p + \iota_p\xi_p\hat{\pi}_{t-1} - \xi_p\hat{\pi}_t$$
$$+\iota_p\xi_p\frac{G'(1)}{G''(1)}\hat{\pi}_{t-1} - \xi_p\frac{G'(1)}{G''(1)}\hat{\pi}_t + \xi_p\frac{1}{G''(1)}\mathrm{d}\tau_t^p$$
$$0 = (1-\xi_p)\left[\hat{\tilde{p}}_t + \frac{G'(1)}{G''(1)}\hat{\tilde{p}}_t\right] + \xi_p\left[\iota_p\hat{\pi}_{t-1} - \hat{\pi}_t + \iota_p\frac{G'(1)}{G''(1)}\hat{\pi}_{t-1} - \frac{G'(1)}{G''(1)}\hat{\pi}_t\right] + \frac{1}{G''(1)}\mathrm{d}\tau_t^p$$

此处去掉 $\frac{1}{G''(1)}\mathrm{d}\tau_t^p$，具体原因还没弄清楚，待进一步观察和计算。
两边同除 $(1+\frac{G'(1)}{G''(1)})$:

$$0 = (1-\xi_p)\hat{\tilde{p}}_t + \xi_p\left[\iota_p\hat{\pi}_{t-1} - \hat{\pi}_t\right]$$

化简可得到:

$$\hat{\tilde{p}}_t = \frac{\xi_p}{1-\xi_p}\left[\hat{\pi}_t - \iota_p\hat{\pi}_{t-1}\right]$$

第二步:现在从方程(27)去趋势的中间品厂商最优价格设定的一阶条件出发,即 $E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}} - \frac{\eta_{t+s}^p\left(\tilde{p}_{i,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^t\pi_{t+l}};\varepsilon_{t+s}^p\right)}{\eta_{t+s}^p(\cdot)-1}mc_{t+s}\right] = 0$ ,

对其进行对数线性化:

1. 两边取全微分:

$$E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\left[\frac{1}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}} - \frac{\eta_{t+s}^p(\cdot)}{\eta_{t+s}^p(\cdot)-1}mc_{t+s}\right]\mathrm{d}\zeta_{t+s} -\right.$$
$$\frac{\zeta_{t+s}}{\zeta_t^2}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}} - \frac{\eta_{t+s}^p(\cdot)}{\eta_{t+s}^p(\cdot)-1}mc_{t+s}\right]\mathrm{d}\zeta_t +$$
$$\frac{\zeta_{t+s}}{\zeta_t}\left(\eta_{t+s}^p(\cdot)-1\right)\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}} - \frac{\eta_{t+s}^p(\cdot)}{\eta_{t+s}^p(\cdot)-1}mc_{t+s}\right]\mathrm{d}y_{i,t+s} +$$
$$\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}} - \frac{\eta_{t+s}^p(\cdot)}{\eta_{t+s}^p(\cdot)-1}mc_{t+s}\right]\mathrm{d}\eta_{t+s}^p(\cdot) +$$
$$\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)\left[\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}}\right]\mathrm{d}\tilde{p}_{i,t} +$$
$$\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)\tilde{p}_{i,t}\mathrm{d}\left(\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^s\pi_{t+l}}\right) -$$
$$\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)\frac{\eta_{t+s}^p(\cdot)}{\eta_{t+s}^p(\cdot)-1}\mathrm{d}mc_{t+s} -$$
$$\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^p(\cdot)-1\right)mc_{t+s}\mathrm{d}\left(\frac{\eta_{t+s}^p(\cdot)}{\eta_{t+s}^p(\cdot)-1}\right) = 0$$

2. 除 d 内的其他式子都在稳态处取值:

由于在稳态时，$\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{s}\pi_{t+l}}-\frac{\eta_{t+s}^{p}(\cdot)}{\eta_{t+s}^{p}(\cdot)-1}mc_{t+s}\right]=1-\frac{\eta^{p}(\cdot)}{\eta^{p}(\cdot)-1}mc$，由于 $\frac{\eta^{p}(\cdot)}{\eta^{p}(\cdot)-1}=markup$，$1-\frac{\eta^{p}(\cdot)}{\eta^{p}(\cdot)-1}mc=0$，因此上面式子的前四项包含 $\left[\tilde{p}_{i,t}\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{s}\pi_{t+l}}-\frac{\eta_{t+s}^{p}(\cdot)}{\eta_{t+s}^{p}(\cdot)-1}mc_{t+s}\right]$，其在稳态时都为 0，因此前四项都为 0。

$$
\begin{aligned}
\mathrm{d}\left(\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{s}\pi_{t+l}}\right)&=\iota_p\frac{\pi^{\iota_p-1}\pi^{1-\iota_p}}{\pi}d\left(\pi_t\right)+\iota_p\frac{\pi^{\iota_p-1}\pi^{1-\iota_p}}{\pi}d\left(\pi_{t+1}\right)+\ldots\\
&-\frac{\left(\pi^{\iota_p}\pi^{1-\iota_p}\right)}{\left(\pi\right)^2}d\left(\pi_{t+1}\right)-\frac{\left(\pi^{\iota_p}\pi^{1-\iota_p}\right)}{\left(\pi\right)^2}d\left(\pi_{t+2}\right)-\ldots\\
&=\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}
\end{aligned}
$$

$$
\mathrm{d}\left(\frac{\eta_{t+s}^{p}\left(\cdot\right)}{\eta_{t+s}^{p}(\cdot)-1}\right)=-\frac{1}{(\eta_{t+s}^{p}(\cdot)-1)^2}\mathrm{d}\eta_{t+s}^{p}\left(\tilde{p}_{i,t}\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{t}\pi_{t+l}};\varepsilon_{t+s}^{p}\right)
$$

$$
=\frac{\eta'}{(\eta_{t+s}^{p}(\cdot)-1)^2}\left[\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{t}\pi_{t+l}}\mathrm{d}\tilde{p}_{i,t}+\tilde{p}_{i,t}\mathrm{d}\left(\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{t}\pi_{t+l}}\right)+\varepsilon_{t+s}^{p}\right]
$$

结合上述分析，可简化以下式子：

$$
\begin{aligned}
0=E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}&\left[\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^{p}(\cdot)-1\right)\left[\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{s}\pi_{t+l}}\right]\mathrm{d}\tilde{p}_{i,t}+\right.\\
&\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^{p}(\cdot)-1\right)\tilde{p}_{i,t}\mathrm{d}\left(\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{s}\pi_{t+l}}\right)-\\
&\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^{p}(\cdot)-1\right)\frac{\eta_{t+s}^{p}\left(\cdot\right)}{\eta_{t+s}^{p}(\cdot)-1}\mathrm{d}mc_{t+s}-\\
&\left.\frac{\zeta_{t+s}}{\zeta_t}y_{i,t+s}\left(\eta_{t+s}^{p}(\cdot)-1\right)mc_{t+s}\mathrm{d}\left(\frac{\eta_{t+s}^{p}\left(\cdot\right)}{\eta_{t+s}^{p}(\cdot)-1}\right)\right]\\
&=E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\frac{\zeta}{\zeta}y_i\left(\eta^{p}(\cdot)-1\right)\left[\hat{\tilde{p}}_{i,t}+\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}+\hat{m}c_{t+s}+\right.\\
&\left.\frac{\eta' mc}{(\eta_{t+s}^{p}(\cdot)-1)^2}\left[\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{t}\pi_{t+l}}\mathrm{d}\tilde{p}_{i,t}+\tilde{p}_{i,t}\mathrm{d}\left(\frac{\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi^{1-\iota_p}}{\prod_{l=1}^{t}\pi_{t+l}}\right)+\varepsilon_{t+s}^{p}\right]\right]\\
&=E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\left[\hat{\tilde{p}}_{i,t}+\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}+\hat{m}c_{t+s}+\right.\\
&\left.\frac{\eta'\cdot mc}{(\eta^{p}(\cdot)-1)^2}\left[\hat{\tilde{p}}_{i,t}+\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}+\varepsilon_{t+s}^{p}\right]\right]\\
&=E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\left[\left(1+\frac{mc\cdot\eta'}{(\eta^{p}(\cdot)-1)^2}\right)\left(\hat{\tilde{p}}_{i,t}+\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}\right)+\right.\\
&\left.\frac{\eta'\cdot mc}{(\eta^{p}(\cdot)-1)^2}\varepsilon_{t+s}^{p}-\hat{m}c_{t+s}\right]
\end{aligned}
$$

注意，$\frac{\eta^{p}(\cdot)}{\eta^{p}(\cdot)-1}=\frac{1}{mc}$。$\frac{\zeta}{\zeta}y_i\left(\eta^{p}(\cdot)-1\right)$ 与 $s$ 无关，是常数，两边相除常数，可去掉常数项。

定义 $\phi_p=\frac{\eta^p}{\eta^p-1}$，为价格 markup，可得 $\phi_p-1=\frac{1}{\eta_p-1}$，$\frac{\eta'\cdot mc}{(\eta^{p}(\cdot)-1)^2}=\frac{1}{\eta^{p}(\cdot)-1}\left(\frac{\eta^{p}(\cdot)}{\eta^{p}(\cdot)-1}\cdot mc\right)\frac{\eta'}{\eta}$ =$(\phi_p-1)\cdot 1\cdot\epsilon_p$，定义 $\frac{\eta'}{\eta}=\epsilon_p$，与此同时，与 $s$ 无关的项可加总，$\sum_{s=0}^{\infty}x^s=\frac{1}{1-x}$。

将上式中的最后一项中的 $\hat{\tilde{p}}_{i,t}$ 的项移到左侧，并进行替代：

$$-\frac{1}{1-\xi_p\beta\gamma^{1-\sigma_c}}(1+(\phi_p-1)\epsilon_p)\hat{\tilde{p}}_{i,t} = E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\left[(1+(\phi_p-1)\epsilon_p)\left(\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}\right)+\right.$$

$$\left.\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}\varepsilon_{t+s}^p-\hat{m}c_{t+s}\right]$$

$$\hat{\tilde{p}}_{i,t} = -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s=0}^{\infty}\xi_p^s\beta^s\gamma^{(1-\sigma_c)s}\left[\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+l}+\right.$$

$$\left.\frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+s}^p-\frac{\hat{m}c_{t+s}}{1+(\phi_p-1)\epsilon_p}\right]$$

向前迭代一期，并乘以 $\xi_p\beta\gamma^{1-\sigma_c}$

$$\xi_p\beta\gamma^{1-\sigma_c}\hat{\tilde{p}}_{i,t+1} = -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s=0}^{\infty}\xi_p^{s+1}\beta^{s+1}\gamma^{(1-\sigma_c)(s+1)}\left[\sum_{l=1}^{s}\iota_p\hat{\pi}_{t+1+l-1}-\sum_{l=1}^{s}\hat{\pi}_{t+1+l}+\right.$$

$$\left.\frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+1+s}^p-\frac{\hat{m}c_{t+1+s}}{1+(\phi_p-1)\epsilon_p}\right]$$

令 $s+1=s^*, l+1=l^*$，上式可变为

$$\xi_p\beta\gamma^{1-\sigma_c}E_t\hat{\tilde{p}}_{i,t+1} = -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=1}^{\infty}\xi_p^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=2}^{s^*}\iota_p\hat{\pi}_{t+l^*-1} - \sum_{l^*=2}^{s^*}\hat{\pi}_{t+l^*} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+s^*}^p - \frac{\hat{m}c_{t+s^*}}{1+(\phi_p-1)\epsilon_p}\right]$$

$$= -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=1}^{\infty}\xi_p^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_p\hat{\pi}_{t+l^*-1} - \sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*} - \iota_p\hat{\pi}_t + \hat{\pi}_{t+1} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+s^*}^p - \frac{\hat{m}c_{t+s^*}}{1+(\phi_p-1)\epsilon_p}\right]$$

$$= -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=0}^{\infty}\xi_p^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_p\hat{\pi}_{t+l^*-1} - \sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*} - \iota_p\hat{\pi}_t + \hat{\pi}_{t+1} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+s^*}^p - \frac{\hat{m}c_{t+s^*}}{1+(\phi_p-1)\epsilon_p}\right] + (1-\xi_p\beta\gamma^{1-\sigma_c})\left[-\iota_p\hat{\pi}_t + \hat{\pi}_{t+1} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right]$$

$$= -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=0}^{\infty}\xi_p^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_p\hat{\pi}_{t+l^*-1} - \sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+s^*}^p - \frac{\hat{m}c_{t+s^*}}{1+(\phi_p-1)\epsilon_p}\right] + (1-\xi_p\beta\gamma^{1-\sigma_c})\left[-\iota_p\hat{\pi}_t + \hat{\pi}_{t+1} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right] - (-\iota_p\hat{\pi}_t + \hat{\pi}_{t+1})$$

$$= -(1-\xi_p\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=0}^{\infty}\xi_p^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_p\hat{\pi}_{t+l^*-1} - \sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*} + \frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_{t+s^*}^p - \frac{\hat{m}c_{t+s^*}}{1+(\phi_p-1)\epsilon_p}\right] + (-\xi_p\beta\gamma^{1-\sigma_c})(-\iota_p\hat{\pi}_t + \hat{\pi}_{t+1}) + (1-\xi_p\beta\gamma^{1-\sigma_c})\left[\frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right]$$

使用 $\hat{\tilde{p}}_{i,t} - \xi_p\beta\gamma^{1-\sigma_c}E_t\hat{\tilde{p}}_{i,t+1}$ 可得：

$$\hat{\tilde{p}}_{i,t} - \xi_p\beta\gamma^{1-\sigma_c}E_t\hat{\tilde{p}}_{i,t+1} = (\xi_p\beta\gamma^{1-\sigma_c})(-\iota_p\hat{\pi}_t + \hat{\pi}_{t+1}) - (1-\xi_p\beta\gamma^{1-\sigma_c})\left[\frac{\frac{\eta'\cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right]$$

利用（23）式的线性化结果 $\hat{p}_t = \frac{\xi_p}{1-\xi_p}\left[\hat{\pi}_t - \iota_p \hat{\pi}_{t-1}\right]$ 替换上式：

$$\frac{\xi_p}{1-\xi_p}\left(\hat{\pi}_t - \iota_p \hat{\pi}_{t-1}\right) - \xi_p \beta \gamma^{1-\sigma_c}\left[\frac{\xi_p}{1-\xi_p}\left(\hat{\pi}_{t+1} - \iota_p \hat{\pi}_t\right)\right]$$
$$= \left(\xi_p \beta \gamma^{1-\sigma_c}\right)\left(-\iota_p \hat{\pi}_t + \hat{\pi}_{t+1}\right) - \left(1 - \xi_p \beta \gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta' \cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right]$$

将上式中 $\hat{\pi}_t$ 项都移到左边：

$$\frac{\xi_p}{1-\xi_p}\hat{\pi}_t - \frac{\xi_p}{1-\xi_p}\iota_p \hat{\pi}_{t-1} - \xi_p \beta \gamma^{1-\sigma_c}\frac{\xi_p}{1-\xi_p}\hat{\pi}_{t+1} + \xi_p \beta \gamma^{1-\sigma_c}\frac{\xi_p}{1-\xi_p}\iota_p \hat{\pi}_t + \xi_p \beta \gamma^{1-\sigma_c}\iota_p \hat{\pi}_t =$$
$$\xi_p \beta \gamma^{1-\sigma_c}\hat{\pi}_{t+1} - \left(1 - \xi_p \beta \gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta' \cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right]$$

$$\frac{\xi_p}{1-\xi_p}\hat{\pi}_t + \xi_p \beta \gamma^{1-\sigma_c}\frac{\xi_p}{1-\xi_p}\iota_p \hat{\pi}_t + \xi_p \beta \gamma^{1-\sigma_c}\iota_p \hat{\pi}_t = \xi_p \beta \gamma^{1-\sigma_c}\frac{\xi_p}{1-\xi_p}\hat{\pi}_{t+1} +$$
$$\frac{\xi_p}{1-\xi_p}\iota_p \hat{\pi}_{t-1} + \xi_p \beta \gamma^{1-\sigma_c}\hat{\pi}_{t+1} - \left(1 - \xi_p \beta \gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta' \cdot mc}{(\eta^p(\cdot)-1)^2}}{1+(\phi_p-1)\epsilon_p}\varepsilon_t^p - \frac{\hat{m}c_t}{1+(\phi_p-1)\epsilon_p}\right]$$

$$\frac{\xi_p}{1-\xi_p}\beta \gamma^{1-\sigma_c}\iota_p \hat{\pi}_t + \frac{\xi_p}{1-\xi_p}\hat{\pi}_t = \frac{\xi_p}{1-\xi_p}(1 + \beta \gamma^{1-\sigma_c}\iota_p)\hat{\pi}_t$$

同除 $\frac{\xi_p}{1-\xi_p}(1+\beta\gamma^{1-\sigma_c}\iota_p)$ 可得

$$\begin{aligned}\hat{\pi}_t =& \frac{\iota_p}{1+\beta\gamma^{1-\sigma_c}\iota_p}\hat{\pi}_{t-1} + \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p}E\hat{\pi}_{t+1} \\ & - \frac{1-\xi_p\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p}\frac{1-\xi_p}{\xi_p}\frac{1}{1+(\phi_p-1)\,\varepsilon_p}\mu_t^p + \varepsilon_t^p\end{aligned}$$

其中，$-\hat{m}c_t = \mu_t^p$，注意，此处将冲击项的系数标准化为 1。如果其他地方出现 $\varepsilon_t^p$，其系数应该乘以此处系数的倒数。

## 2.11 SW(11): 资本收益率方程的对数线性化

从方程（25）去趋势的中间品厂商成本最小化的一阶条件 $k_t^s = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}L_t$ 可得：

1. 对方程（25）取对数：

$$\log k_t^s = \log\frac{\alpha}{1-\alpha} + \log w_t + \log L_t - \log r_t^k$$

2. 上式在稳态处的取值：

$$\log k^s = \log\frac{\alpha}{1-\alpha} + \log w + \log L - \log r^k$$

3. 两式相减：

$$\begin{aligned}&\log k_t^s - \log k^s = \log w_t - \log w + \log L_t - \log L - \log r_t^k - \log r^k \\ &\Longrightarrow \hat{k}_t^s = \hat{w}_t + \hat{l}_t - \hat{r}_t^k \\ &\Longrightarrow \hat{r}_t^k = -(\hat{k}_t^s - \hat{l}_t) + \hat{w}_t\end{aligned}$$

### 2.12 SW(12): 工资 markup 的对数线性化

从方程（31）去趋势的劳动一阶条件 $w_t^h = \left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)\bar{L}_t^{\sigma_l}$ 出发：

1. 对方程（31）取全微分：

$$\mathrm{d}w_t^h = \sigma_l\left(c_t - \frac{\lambda}{\gamma}c_{t-1}\right)\bar{L}_t^{\sigma_l - 1}\mathrm{d}\bar{L} + \bar{L}_t^{\sigma_l}\left(\mathrm{d}c_t - \frac{\lambda}{\gamma}\mathrm{d}c_{t-1}\right)$$

2. 方程（31）在稳态处的取值：

$$w^h = \left(c - \frac{\lambda}{\gamma}c\right)\bar{L}^{\sigma_l}$$

3. 微分方程除以稳态值：

$$\frac{\mathrm{d}w_t^h}{w^h} = \sigma_l\frac{\mathrm{d}\bar{L}_t}{\bar{L}} + \left(1 - \frac{\lambda}{\gamma}\right)^{-1}\frac{\mathrm{d}c_t}{c} - \frac{\frac{\lambda}{\gamma}}{\left(1 - \frac{\lambda}{\gamma}\right)}\frac{\mathrm{d}c_{t-1}}{c} \Longrightarrow \hat{w}_t^h = \sigma_l\hat{l}_t + \frac{1}{\left(1 - \frac{\lambda}{\gamma}\right)}\hat{c}_t - \frac{\frac{\lambda}{\gamma}}{\left(1 - \frac{\lambda}{\gamma}\right)}\hat{c}_{t-1}$$

根据上述结果计算 $\mu_t^w \equiv \hat{w}_t - \hat{w}_t^h$

$$\mu_t^w \equiv \hat{w}_t - \hat{w}_t^h = \hat{w}_t - \left(\sigma_l\hat{l}_t + \frac{1}{1 - \frac{\lambda}{\gamma}}\left(\hat{c}_t - \frac{\lambda}{\gamma}\hat{c}_{t-1}\right)\right)$$

### 2.13 SW(13): 工资菲利普斯曲线的推导

第一步：从方程 (37) 方程去趋势的总工资方程 $w_t = (1-\xi_w)\tilde{w}_t H'^{-1}\left[\frac{\tilde{w}_t}{w_t}\tau_t^w\right] + \xi_w \pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]$ 出发，对其进行对数线性化：

1. 两边取全微分：

$$\begin{aligned}
\mathrm{d}w_t = {} & (1-\xi_w)H'^{-1}\left[\frac{\tilde{w}_t}{w_t}\tau_t^w\right]\mathrm{d}\tilde{w}_t + (1-\xi_w)\frac{\tilde{w}_t}{w_t}\frac{\tau_t^w}{H''(l_{i,t})}\mathrm{d}\tilde{w}_t - (1-\xi_w)\tilde{w}_t\frac{\tilde{w}_t\tau_t^w}{w_t^2 H''(l_{i,t})}\mathrm{d}w_t \\
& + \iota_w\xi_w\pi_{t-1}^{\iota_w - 1}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]\mathrm{d}\pi_{t-1} \\
& - \xi_w\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-2}w_{t-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]\mathrm{d}\pi_t \\
& + \xi_w\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]\mathrm{d}w_{t-1} \\
& + \xi_w\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}\frac{1}{H''(l_{i,t})}\left[\iota_w\frac{\pi_{t-1}^{\iota_w - 1}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\mathrm{d}\pi_{t-1}\right. \\
& \left. - \frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-2}w_{t-1}}{w_t}\tau_t^w\mathrm{d}\pi_t + \frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}}{w_t}\tau_t^w\mathrm{d}w_{t-1} - \frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t^2}\tau_t^w\mathrm{d}w_t\right]
\end{aligned}$$

2. 对处 d 内之外的其他表达式在稳态处取值：

注意，在稳态值处，有 $H'^{-1}(n^*) = l = 1$，其中 $l_{i,t} = \frac{L_{i,t}}{L_t}$ ，$n_{i,t} = \frac{\tilde{w}_t}{w_t}\tau_t^w$ ，$\tau^w = H'(1)$，因此在稳态处有：

$$(1-\xi_w)H'^{-1}\left[\frac{\tilde{w}_t}{w_t}\tau_t^w\right]\mathrm{d}\tilde{w}_t = (1-\xi_w)\mathrm{d}\tilde{w}_t$$

$$(1-\xi_w)\frac{\tilde{w}_t}{w_t}\frac{\tau_t^w}{H''^{(l_{i,t})}}\mathrm{d}\tilde{w}_t=(1-\xi_w)\frac{\tau^w}{H''(1)}\mathrm{d}\tilde{w}_t=(1-\xi_w)\frac{H'(1)}{H''(1)}\mathrm{d}\tilde{w}_t$$

$$(1-\xi_w)\tilde{w}_t\frac{\tilde{w}_t\tau_t^w}{w_t^2H''(l_{i,t})}\mathrm{d}w_t=(1-\xi_w)\frac{H'(1)}{H''(1)}\mathrm{d}w_t$$

$$\iota_w\xi_w\pi_{t-1}^{\iota_w-1}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]\mathrm{d}\pi_{t-1}=$$
$$\iota_w\xi_w wH'^{-1}(H'(1))\frac{\mathrm{d}\pi_{t-1}}{\pi}=\iota_w\xi_w w\hat{\pi}_{t-1}$$

其中，$H'^{-1}\left(H'(1)\right)=1$

$$-\xi_w\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-2}w_{t-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]\mathrm{d}\pi_t=-\xi_w w\hat{\pi}_t$$

$$\xi_w\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}H'^{-1}\left[\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\right]\mathrm{d}w_{t-1}=\xi_w\mathrm{d}w_{t-1}$$

$$\begin{aligned}&\xi_w\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}\frac{1}{H''(l_{i,t})}\Bigg[\iota_w\frac{\pi_{t-1}^{\iota_w-1}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t}\tau_t^w\mathrm{d}\pi_{t-1}\\&-\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-2}w_{t-1}}{w_t}\tau_t^w\mathrm{d}\pi_t+\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}}{w_t}\tau_t^w\mathrm{d}w_{t-1}-\frac{\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}\pi_t^{-1}w_{t-1}}{w_t^2}\tau_t^w\mathrm{d}w_t\Bigg]\\&=\xi_w w\frac{1}{H''(1)}\left[\iota_wH'(1)\hat{\pi}_{t-1}-H'(1)\hat{\pi}_t+H'(1)\hat{w}_{t-1}-H'(1)\hat{w}_t\right]\\&=\xi_w w\frac{H'(1)}{H''(1)}\left[\iota_w\hat{\pi}_{t-1}-\hat{\pi}_t+\hat{w}_{t-1}-\hat{w}_t\right]\end{aligned}$$

综合以上式子可得：

$$\begin{aligned}\mathrm{d}w_t=&(1-\xi_w)\mathrm{d}\tilde{w}_t+(1-\xi_w)\frac{H'(1)}{H''(1)}\mathrm{d}\tilde{w}_t-(1-\xi_w)\frac{H'(1)}{H''(1)}\mathrm{d}w_t\\&+\iota_w\xi_w w\hat{\pi}_{t-1}-\xi_w w\hat{\pi}_t+\xi_w\mathrm{d}w_{t-1}+\xi_w w\frac{H'(1)}{H''(1)}\left[\iota_w\hat{\pi}_{t-1}-\hat{\pi}_t+\hat{w}_{t-1}-\hat{w}_t\right]\end{aligned}$$

两边同除 $w$：

$$\begin{aligned}\hat{w}_t=&(1-\xi_w)\hat{\tilde{w}}_t+(1-\xi_w)\frac{H'(1)}{H''(1)}\hat{\tilde{w}}_t-(1-\xi_w)\frac{H'(1)}{H''(1)}\hat{w}_t\\&+\iota_w\xi_w\hat{\pi}_{t-1}-\xi_w\hat{\pi}_t+\xi_w\hat{w}_{t-1}+\xi_w\frac{H'(1)}{H''(1)}\left[\iota_w\hat{\pi}_{t-1}-\hat{\pi}_t+\hat{w}_{t-1}-\hat{w}_t\right]\end{aligned}$$

合并 $\hat{\tilde{w}}_t$ 项：

$$\begin{aligned}(1-\xi_w)\left(1+\frac{H'(1)}{H''(1)}\right)\hat{\tilde{w}}_t=&\left(1+\frac{H'(1)}{H''(1)}\right)\hat{w}_t-\iota_w\xi_w\left(1+\frac{H'(1)}{H''(1)}\right)\hat{\pi}_{t-1}\\&+\xi_w\left(1+\frac{H'(1)}{H''(1)}\right)\hat{\pi}_t-\xi_w\left(1+\frac{H'(1)}{H''(1)}\right)\hat{w}_{t-1}\end{aligned}$$

两边同除 $(1-\xi_w)\left(1+\frac{H'(1)}{H''(1)}\right)$ 可得：

$$\hat{\tilde{w}}_t=\frac{1}{1-\xi_w}\hat{w}_t-\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}+\frac{\xi_w}{1-\xi_w}\hat{\pi}_t-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}$$

第二步：现在从方程（36）去趋势的工会设定最优劳动工资的一阶条件 $E_t\sum_{s=0}^{\infty}\xi_w^s\beta^s$ $\gamma^{(1-\sigma_c)s}\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w\left(\frac{\tilde{w}_{l,t}}{w_{t+s}}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{i=1}^t\pi_{t+l}};\varepsilon_{t+s}^w\right)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]=0$ 出发，对其进行对数线性化：

1. 两边取全微分：

$$\begin{aligned}\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\Bigg[&\frac{1}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]\mathrm{d}\zeta_{t+s}-\\&\frac{\zeta_{t+s}}{\zeta_t^2}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]\mathrm{d}\zeta_t+\\&\frac{\zeta_{t+s}}{\zeta_t}\left(\eta_{t+s}^w(\cdot)-1\right)\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]\mathrm{d}L_{l,t+s}+\\&\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]\mathrm{d}\eta_{t+s}^w(\cdot)+\\&\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\left[\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}\right]\mathrm{d}\tilde{w}_{l,t}+\\&\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\tilde{w}_{i,t}\mathrm{d}\left(\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}\right)-\\&\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}\mathrm{d}w_{t+s}^h-\\&\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)w_{t+s}^h\mathrm{d}\left(\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}\right)=0\end{aligned}$$

2. 除 d 内其他式子都在稳态处取值：由于在稳态时，$\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]=\tilde{w}_l-\frac{\eta^w(\cdot)}{\eta^w(\cdot)-1}w^h=0$，上式的前四项都包含 $\left[\tilde{w}_{l,t}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}-\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}w_{t+s}^h\right]$，其在稳态时都为 0，因此前四项都为 0。

$$\begin{aligned}\mathrm{d}\left(\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}\right)&=\iota_w\frac{\pi^{\iota_w-1}\pi^{1-\iota_w}}{\pi}d\left(\pi_t\right)+\iota_w\frac{\pi^{\iota_w-1}\pi^{1-\iota_w}}{\pi}d\left(\pi_{t+1}\right)+\ldots\\&-\frac{\left(\pi^{\iota_w}\pi^{1-\iota_w}\right)}{(\pi)^2}d\left(\pi_{t+1}\right)-\frac{\left(\pi^{\iota_w}\pi^{1-\iota_w}\right)}{(\pi)^2}d\left(\pi_{t+2}\right)-\ldots\\&=\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}\end{aligned}$$

$$\begin{aligned}\mathrm{d}\left(\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}\right)&=-\frac{1}{(\eta_{t+s}^w(\cdot)-1)^2}\mathrm{d}\eta_{t+s}^w\left(\frac{\tilde{w}_{l,t}}{w_{t+s}}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{i=1}^t\pi_{t+l}};\varepsilon_{t+s}^w\right)\\&=-\frac{\eta'}{(\eta_{t+s}^w(\cdot)-1)^2}\Bigg[\frac{1}{w_{t+s}}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^t\pi_{t+l}}\mathrm{d}\tilde{w}_{l,t}-\\&\frac{\tilde{w}_{l,t}}{w_{t+s}^2}\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^t\pi_{t+l}}\mathrm{d}w_{t+s}+\frac{\tilde{w}_{l,t}}{w_{t+s}}\mathrm{d}\left(\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^t\pi_{t+l}}\right)+\varepsilon_{t+s}^w\Bigg]\end{aligned}$$

结合上述分析，可简化以下式子:

$$
\begin{aligned}
&\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\left[\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\left[\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}\right]\mathrm{d}\tilde{w}_{l,t}+\right.\\
&\qquad\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\tilde{w}_{i,t}\mathrm{d}\left(\frac{\prod_{l=1}^s\pi_{t+l-1}^{\iota_w}\pi^{1-\iota_w}}{\prod_{l=1}^s\pi_{t+l}}\right)-\\
&\qquad\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}\mathrm{d}w_{t+s}^h-\\
&\qquad\left.\frac{\zeta_{t+s}}{\zeta_t}L_{l,t+s}\left(\eta_{t+s}^w(\cdot)-1\right)w_{t+s}^h\mathrm{d}\left(\frac{\eta_{t+s}^w(\cdot)}{\eta_{t+s}^w(\cdot)-1}\right)\right]\\
=&\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\frac{\zeta}{\zeta}L_l\left(\eta^w(\cdot)-1\right)\left[\mathrm{d}\tilde{w}_{l,t}+\tilde{w}_l\left(\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}\right)-\tilde{w}_l\hat{w}_{t+s}^h+\right.\\
&\qquad\left.\frac{\eta' w^h}{(\eta_{t+s}^w(\cdot)-1)^2}\left[\frac{1}{w}\mathrm{d}\tilde{w}_{l,t}-\frac{1}{w}\mathrm{d}w_{t+s}+\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}+\varepsilon_{t+s}^w\right]\right]\\
&=\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\left[\hat{\tilde{w}}_{l,t}+\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}-\hat{w}_{t+s}^h+\right.\\
&\qquad\left.\frac{\eta' w^h}{(\eta_{t+s}^w(\cdot)-1)^2\tilde{w}_l}\left[\hat{\tilde{w}}_{l,t}-\hat{w}_{t+s}+\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}+\varepsilon_{t+s}^w\right]\right]\\
=&\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\left[\left(1+\frac{\eta' w^h}{(\eta_{t+s}^w(\cdot)-1)^2\tilde{w}_l}\right)\left(\hat{\tilde{w}}_{l,t}+\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}\right)+\right.\\
&\qquad\left.\frac{\eta' w^h}{(\eta_{t+s}^w(\cdot)-1)^2\tilde{w}_l}(\varepsilon_{t+s}^w-\hat{w}_{t+s})-\hat{w}_{t+s}^h\right]
\end{aligned}
$$

注意，$\frac{\eta^w(\cdot)}{\eta^w(\cdot)-1}=\frac{\tilde{w}_l}{w^h}$，定义 $\phi_w=\frac{\eta^w(\cdot)}{\eta^w(\cdot)-1}$，为工资 markup，可得 $\phi_w-1=\frac{1}{\eta_w-1}$，$\frac{\eta' w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}=\frac{1}{\eta^w(\cdot)-1}\left(\frac{\eta^w(\cdot)}{\eta^w(\cdot)-1}\cdot\frac{w^h}{\tilde{w}_l}\right)\frac{\eta'}{\eta}=(\phi_w-1)\cdot 1\cdot\epsilon_w$，其中，$\frac{\eta^{w\prime}}{\eta^w}=\epsilon_w$，与此同时，与 $s$ 无关的项可加总，$\sum_{s=0}^{\infty}x^s=\frac{1}{1-x}$。$\frac{\zeta}{\zeta}L_l\left(\eta^w(\cdot)-1\right)$ 与 $s$ 无关，是常数，可两边相除，去掉常数项。
将上式中的 $\hat{\tilde{w}}_{l,t}$ 的项移到左侧，并进行相关替代:

$$
\begin{aligned}
-\frac{1}{1-\xi_w\beta\gamma^{1-\sigma_c}}(1+(\phi_w-1)\epsilon_w)\hat{\tilde{w}}_{l,t}=&\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\left[(1+(\phi_w-1)\epsilon_w)\left(\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}\right)\right.\\
&\qquad\left.\frac{\eta' w^h}{(\eta_{t+s}^w(\cdot)-1)^2\tilde{w}_l}(\varepsilon_{t+s}^w-\hat{w}_{t+s})-\hat{w}_{t+s}^h\right]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\tilde{w}}_{l,t}=-(1-\xi_w\beta\gamma^{1-\sigma_c})&\sum_{s=0}^{\infty}\xi_w^s\beta^s\gamma^{(1-\sigma_c)s}\left[\left(\sum_{l=1}^s\iota_w\hat{\pi}_{t+l-1}-\sum_{l=1}^s\hat{\pi}_{t+l}\right)+\right.\\
&\left.\frac{\frac{\eta' w^h}{(\eta_{t+s}^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon_{t+s}^w-\hat{w}_{t+s})-\frac{\hat{w}_{t+s}^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

向前迭代一期，并乘以 $\xi_w\beta\gamma^{1-\sigma_c}$

$$
\begin{aligned}
\xi_w\beta\gamma^{1-\sigma_c}\hat{\tilde{w}}_{l,t+1}=-(1-\xi_w\beta\gamma^{1-\sigma_c})E_t&\sum_{s=0}^{\infty}\xi_w^{s+1}\beta^{s+1}\gamma^{(1-\sigma_c)(s+1)}\left[\sum_{l=1}^s\iota_w\hat{\pi}_{t+1+l-1}-\sum_{l=1}^s\hat{\pi}_{t+1+l}+\right.\\
&\left.\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon_{t+1+s}^w-\hat{w}_{t+s+1})-\frac{\hat{w}_{t+1+s}^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

令 $s+1=s^*, l+1=l^*$，上式可变为

$$\xi_w\beta\gamma^{1-\sigma_c}E_t\hat{\tilde{w}}_{l,t+1}=-(1-\xi_w\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=1}^{\infty}\xi_w^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=2}^{s^*}\iota_w\hat{\pi}_{t+l^*-1}-\sum_{l^*=2}^{s^*}\hat{\pi}_{t+l^*}+\right.$$
$$\left.\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_{t+s^*}-\hat{w}_{t+s^*})-\frac{\hat{w}^h_{t+s^*}}{1+(\phi_w-1)\epsilon_w}\right]$$
$$=-(1-\xi_w\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=1}^{\infty}\xi_w^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_w\hat{\pi}_{t+l^*-1}-\sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*}-\iota_w\hat{\pi}_t\right.$$
$$\left.+\hat{\pi}_{t+1}+\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_{t+s^*}-\hat{w}_{t+s^*})-\frac{\hat{w}^h_{t+s^*}}{1+(\phi_w-1)\epsilon_w}\right]$$
$$=-(1-\xi_w\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=0}^{\infty}\xi_w^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_w\hat{\pi}_{t+l^*-1}-\sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*}-\iota_w\hat{\pi}_t\right.$$
$$\left.+\hat{\pi}_{t+1}+\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_{t+s^*}-\hat{w}_{t+s^*})-\frac{\hat{w}^h_{t+s^*}}{1+(\phi_w-1)\epsilon_w}\right]+(1-\xi_w\beta\gamma^{1-\sigma_c})$$
$$\left[-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1}+\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_t-\hat{w}_t)-\frac{\hat{w}^h_t}{1+(\phi_w-1)\epsilon_w}\right]$$
$$=-(1-\xi_w\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=0}^{\infty}\xi_w^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_w\hat{\pi}_{t+l^*-1}-\sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*}+\right.$$
$$\left.\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_{t+s^*}-\hat{w}_{t+s^*})-\frac{\hat{w}^h_{t+s^*}}{1+(\phi_w-1)\epsilon_w}\right]+(1-\xi_w\beta\gamma^{1-\sigma_c})\left[-\iota_w\hat{\pi}_t+\right.$$
$$\left.+\hat{\pi}_{t+1}\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_t-\hat{w}_t)-\frac{\hat{w}^h_t}{1+(\phi_w-1)\epsilon_w}\right]-(\iota_w\hat{\pi}_t+\hat{\pi}_{t+1})$$
$$=-(1-\xi_w\beta\gamma^{1-\sigma_c})E_t\sum_{s^*=0}^{\infty}\xi_w^{s^*}\beta^{s^*}\gamma^{(1-\sigma_c)(s^*)}\left[\sum_{l^*=1}^{s^*}\iota_w\hat{\pi}_{t+l^*-1}-\sum_{l^*=1}^{s^*}\hat{\pi}_{t+l^*}+\right.$$
$$\left.\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_{t+s^*}-\hat{w}_{t+s^*})-\frac{\hat{w}^h_{t+s^*}}{1+(\phi_w-1)\epsilon_w}\right]+(-\xi_w\beta\gamma^{1-\sigma_c})$$
$$(-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1})+(1-\xi_w\beta\gamma^{1-\sigma_c})\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_t-\hat{w}_t)-\frac{\hat{w}^h_t}{1+(\phi_w-1)\epsilon_w}\right]$$

使用 $\hat{\tilde{w}}_{l,t}-\xi_w\beta\gamma^{1-\sigma_c}E_t\hat{\tilde{w}}_{l,t+1}$ 可得：

$$\hat{\tilde{w}}_{l,t}-\xi_w\beta\gamma^{1-\sigma_c}E_t\hat{\tilde{w}}_{l,t+1}=(\xi_w\beta\gamma^{1-\sigma_c})(-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1})$$
$$-(1-\xi_w\beta\gamma^{1-\sigma_c})\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon^w_t-\hat{w}_t)-\frac{\hat{w}^h_t}{1+(\phi_w-1)\epsilon_w}\right]$$

利用方程（37）式的线性化结果 $\hat{\tilde{w}}_t=\frac{1}{1-\xi_w}\hat{w}_t-\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}+\frac{\xi_w}{1-\xi_w}\hat{\pi}_t-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}$ 替换上

式：

$$
\begin{aligned}
&\frac{1}{1-\xi_w}\hat{w}_t-\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}+\frac{\xi_w}{1-\xi_w}\hat{\pi}_t-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}-\\
&\xi_w\beta\gamma^{1-\sigma_c}\left(\frac{1}{1-\xi_w}\hat{w}_{t+1}-\frac{\xi_w}{1-\xi_w}\hat{w}_t+\frac{\xi_w}{1-\xi_w}\hat{\pi}_{t+1}-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_t\right)\\
&=\left(\xi_w\beta\gamma^{1-\sigma_c}\right)\left(-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1}\right)\\
&-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}(\varepsilon_t^w-\hat{w}_t)-\frac{\hat{w}_t^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

将上式中 $\hat{w}_t$ 项都移到左边：

$$
\begin{aligned}
&\frac{1}{1-\xi_w}\hat{w}_t+\xi_w\beta\gamma^{1+\sigma_c}\frac{\xi_w}{1-\xi_w}\hat{w}_t-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\frac{(\phi_w-1)\epsilon_w}{1+(\phi_w-1)\epsilon_w}\hat{w}_t=\\
&\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}-\frac{\xi_w}{1-\xi_w}\hat{\pi}_t+\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}+\xi_w\beta\gamma^{1-\sigma_c}\left(\frac{1}{1-\xi_w}\hat{w}_{t+1}+\frac{\xi_w}{1-\xi_w}\hat{\pi}_{t+1}-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_t\right)\\
&+\left(\xi_w\beta\gamma^{1-\sigma_c}\right)\left(-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1}\right)-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}\varepsilon_t^w-\frac{\hat{w}_t^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

利用定义 $\hat{w}_t-\hat{w}_t^h=\mu_t^w$，

$$
\begin{aligned}
&\frac{1}{1-\xi_w}\hat{w}_t+\xi_w\beta\gamma^{1+\sigma_c}\frac{\xi_w}{1-\xi_w}\hat{w}_t-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\frac{(\phi_w-1)\epsilon_w+1}{1+(\phi_w-1)\epsilon_w}\hat{w}_t=\\
&\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}-\frac{\xi_w}{1-\xi_w}\hat{\pi}_t+\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}+\xi_w\beta\gamma^{1-\sigma_c}\left(\frac{1}{1-\xi_w}\hat{w}_{t+1}+\frac{\xi_w}{1-\xi_w}\hat{\pi}_{t+1}-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_t\right)\\
&+\left(\xi_w\beta\gamma^{1-\sigma_c}\right)\left(-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1}\right)-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}\varepsilon_t^w+\frac{\hat{w}_t-\hat{w}_t^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{1-\xi_w}\hat{w}_t+\xi_w\beta\gamma^{1-\sigma_c}\frac{\xi_w}{1-\xi_w}\hat{w}_t-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\hat{w}_t=\\
&\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}-\frac{\xi_w}{1-\xi_w}\hat{\pi}_t+\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}+\xi_w\beta\gamma^{1-\sigma_c}\left(\frac{1}{1-\xi_w}\hat{w}_{t+1}+\frac{\xi_w}{1-\xi_w}\hat{\pi}_{t+1}-\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_t\right)\\
&+\left(\xi_w\beta\gamma^{1-\sigma_c}\right)\left(-\iota_w\hat{\pi}_t+\hat{\pi}_{t+1}\right)-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}\varepsilon_t^w+\frac{\hat{w}_t-\hat{w}_t^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\xi_w}{1-\xi_w}(1+\beta\gamma^{1-\sigma_c})\hat{w}_t=\frac{\xi_w}{1-\xi_w}\hat{w}_{t-1}+\xi_w\beta\gamma^{1-\sigma_c}\frac{1}{1-\xi_w}\hat{w}_{t+1}+\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_{t-1}\\
&-\frac{\xi_w}{1-\xi_w}\hat{\pi}_t-\xi_w\beta\gamma^{1-\sigma_c}\frac{\xi_w}{1-\xi_w}\iota_w\hat{\pi}_t-\xi_w\beta\gamma^{1-\sigma_c}\iota_w\hat{\pi}_t+\xi_w\beta\gamma^{1-\sigma_c}\frac{\xi_w}{1-\xi_w}\hat{\pi}_{t+1}+\xi_w\beta\gamma^{1-\sigma_c}\hat{\pi}_{t+1}\\
&-\left(1-\xi_w\beta\gamma^{1-\sigma_c}\right)\left[\frac{\frac{\eta'\cdot w^h}{(\eta^w(\cdot)-1)^2\tilde{w}_l}}{1+(\phi_w-1)\epsilon_w}\varepsilon_t^w+\frac{\hat{w}_t-\hat{w}_t^h}{1+(\phi_w-1)\epsilon_w}\right]
\end{aligned}
$$

上式两边同除 $\frac{\xi_w}{1-\xi_w}(1+\beta\gamma^{1-\sigma_c})$ 可得：

$$
\begin{aligned}
\hat{w}_t=&\frac{1}{1+\beta\gamma^{1-\sigma_c}}\hat{w}_{t-1}+\frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}}\left(E\hat{w}_{t+1}+E\hat{\pi}_{t+1}\right)-\frac{1+\beta\gamma^{1-\sigma_c}\iota_w}{1+\beta\gamma^{1-\sigma_c}}\hat{\pi}_t+\frac{\iota_w}{1+\beta\gamma^{1-\sigma_c}}\hat{\pi}_{t-1}\\
&-\frac{1-\xi_w\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}}\frac{1-\xi_w}{\xi_w}\frac{1}{1+(\phi_w-1)\,\varepsilon_w}\mu_t^w+\varepsilon_t^w
\end{aligned}
$$

注意，$\hat{w}_t - \hat{w}_t^h = \mu_t^w$ 此处将冲击项的系数标准化为 1。如果其他地方出现 $\varepsilon_t^w$，其系数应该乘以此处系数的倒数。

## 2.14 SW(14): 货币政策方程的对数线性化

从方程（38）去趋势的货币政策方程 $\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho}\left[\left(\frac{\pi_t}{\pi}\right)^{r_\pi}\left(\frac{y_t}{y_t^p}\right)^{r_y}\right]^{1-\rho}\left(\frac{y_t/y_{t-1}}{y_t^p/y_{t-1}^p}\right)^{r_{\Delta y}} \mathrm{e}^{\varepsilon_t^r}$ 出发，对方程（38）取对数：

$$\begin{aligned}
\log\frac{R_t}{R} =& \rho\log\left(\frac{R_{t-1}}{R}\right) + (1-\rho)\left[r_\pi\log\left(\frac{\pi_t}{\pi}\right) + r_y\log\left(\frac{y_t}{y_t^p}\right)\right] + r_{\Delta y}\log\left(\frac{y_t/y_{t-1}}{y_t^p/y_{t-1}^p}\right) + \varepsilon_t^r\\
\log\frac{R_t}{R} =& \rho\log\left(\frac{R_{t-1}}{R}\right) + (1-\rho)\left[r_\pi\log\left(\frac{\pi_t}{\pi}\right) + r_y\log\left(\frac{y_t}{y}\frac{y}{y_t^p}\right)\right] + r_{\Delta y}\log\left(\frac{y_t}{y}\frac{y}{y_{t-1}}\frac{y_{t-1}^p}{y}\frac{y}{y_t^p}\right) + \varepsilon_t^r\\
\hat{r}_t =& \rho\hat{r}_{t-1} + (1-\rho)\left[r_\pi\hat{\pi}_t + r_y(\hat{y}_t - \hat{y}_t^p)\right] + r_{\Delta y}(\hat{y}_t - \hat{y}_t^p - \hat{y}_{t-1} + \hat{y}_{t-1}^p) + \varepsilon_t^r
\end{aligned}$$